Single and multiple linear regression analysis

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Outline of the session

- Introduction
- Simple linear regression analysis
- SPSS example of simple linear regression analysis
- Additional topics in multiple linear regression analysis
  - Adjusted R-squared
  - Standardised regression coefficients
  - Multicollinearity
  - A note on categorical predictors (dummy variables)

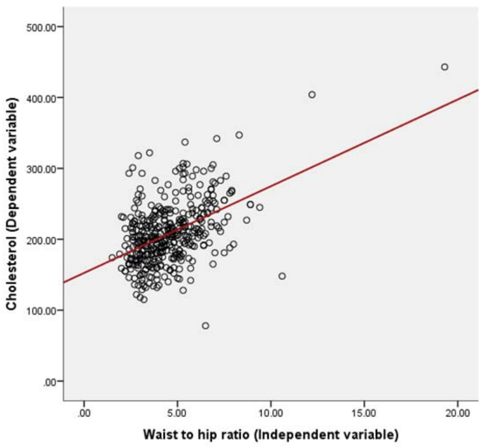
Introduction: Simple linear regression (1)

- Suppose we collect data on two variables:
  - Waist to hip ratio (X).
  - Cholesterol (Y).
- For each participant we now have a pair of observations (X_i, Y_i).

ID	Cholesterol (Y)	Ratio (X)
1	203	3.60
2	165	6.90
3	228	6.20
4	78	6.50
5	249	8.90
⋮	⋮	⋮

Introduction: Simple linear regression (2)

- We want to fit a straight line that describes the linear relationship between X and Y.

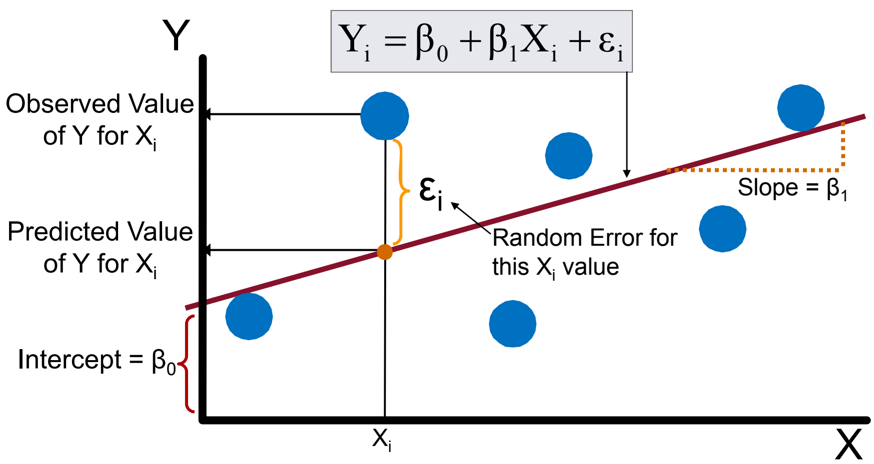


Introduction: Simple linear regression (3)

- Simple linear regression is a technique that is used to explore the nature of the relationship between two variables.
- Regression analysis enables us to investigate the change in one variable, called the response (dependent variable), which corresponds to a given change in the other, known as the explanatory variable (independent variable).
- The ultimate objective of regression analysis is to predict or estimate the value of the response that is associated with a fixed value of the explanatory variable.

Simple Linear Regression Model (1)

Population model



Simple Linear Regression Model (2)

Population model

Population Y intercept

Population slope coefficient

Independent variable

Random error term

Dependent variable

$$\underbrace{Y_i = \beta_0 + \beta_1 X_i}_{\text{Linear component}} + \underbrace{\varepsilon_i}_{\text{Random error component}}$$

Simple Linear Regression Equation (Prediction Line)

The simple linear regression equation provides an estimate of the population regression line

Estimated (or predicted) Y value for observation i

Estimate of the regression intercept

Estimate of the regression slope

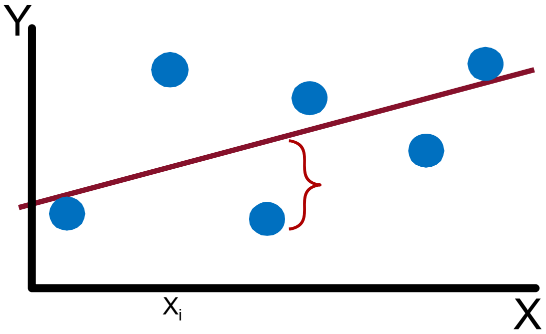
Value of X for observation i

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$$

Method of least squares

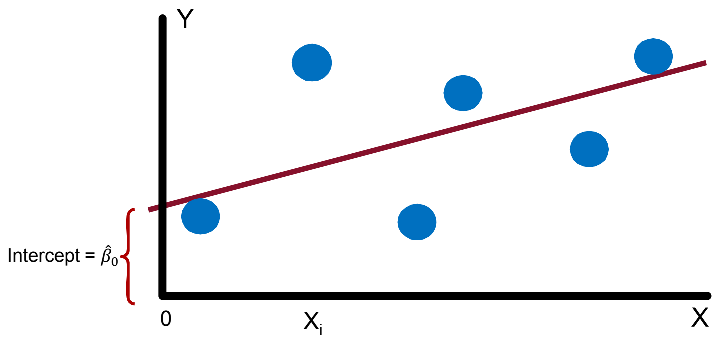
$\hat{\beta}_0$ and $\hat{\beta}_1$ are obtained by finding the values of that **minimize the sum of the squared differences** between Y and $\hat{Y}$:

$$\min \sum (Y_i - \hat{Y}_i)$$



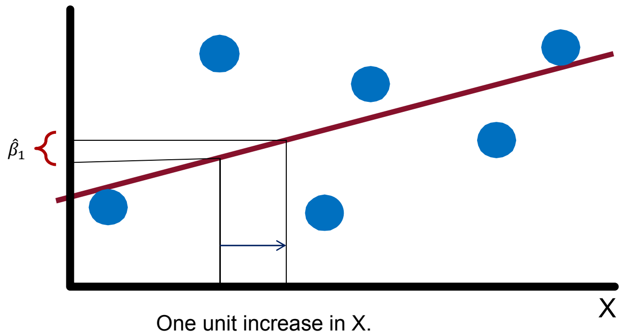
Interpretation of intercept term

- $\hat{\beta}_0$ is the estimated average values of Y when the value of X is zero.
- $\hat{\beta}_0$ has only practical application if X=0 is in the range of observed X values.



Interpretation of slope term

- $\hat{\beta}_1$ estimates the change in the average value of Y as a result of a one-unit increase in X.



Coefficient of Determination (R^2)

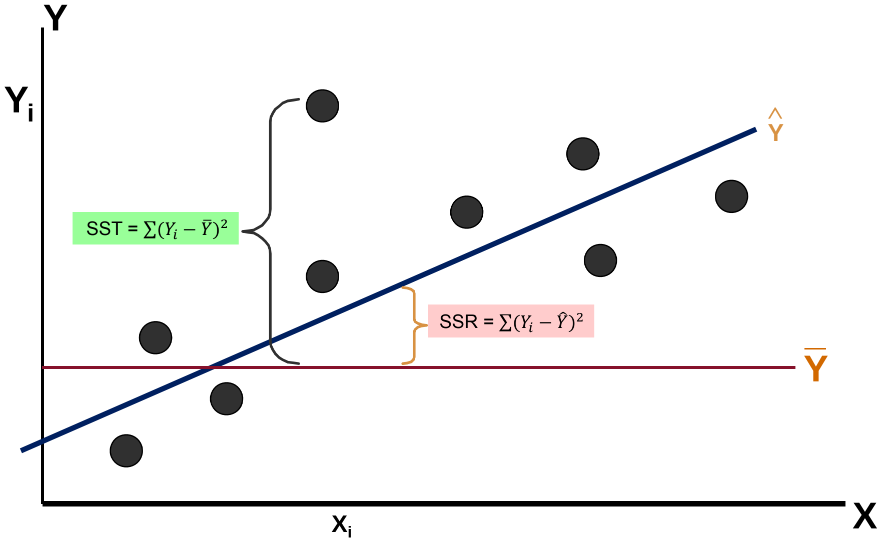
- The coefficient of determination is the portion of the total variation in the dependent variable that is explained by variation in the independent variable

- The coefficient of determination is also called R-squared and is denoted as R^2.

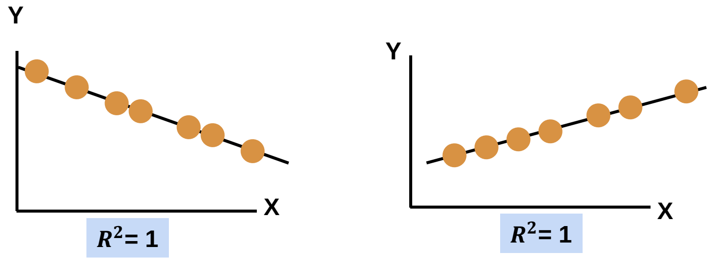
$$R^2 = \frac{SSR}{SST} = \frac{\text{regression sum of squares}}{\text{total sum of squares}}$$

$$0 \leq R^2 \leq 1$$

Coefficient of Determination (R^2)

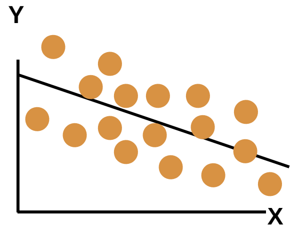


Examples of approximate R^2 values (1)

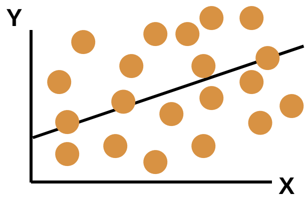


Perfect linear relationship between X and Y.
100% of the variation in Y is explained by variation in X.

Examples of approximate R^2 values (2)

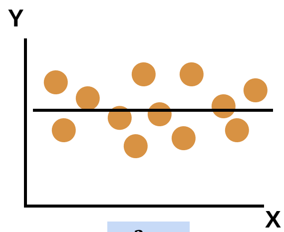


$0 < R^2 < 1$



Weak linear relationships between X and Y.
Some but not all of the variation in Y is explained by variation in X.

Examples of Approximate R^2 values (3)



$R^2 = 0$

No linear relationship between X and Y.
The value of Y does not depend on X.
None of the variation in Y is explained by the variation in X.

Assumption of simple linear regression models

- Linearity
  - The relationship between X and Y is linear.
- Independence of errors
  - Error values are statistically independent.
- Normality of error
  - Error values are normally distributed for any given value of X.
- Equal variance (homoscedasticity)
  - The probability distribution of the errors has constant variance.

Test assumptions after fitting the model by making use of the residuals.

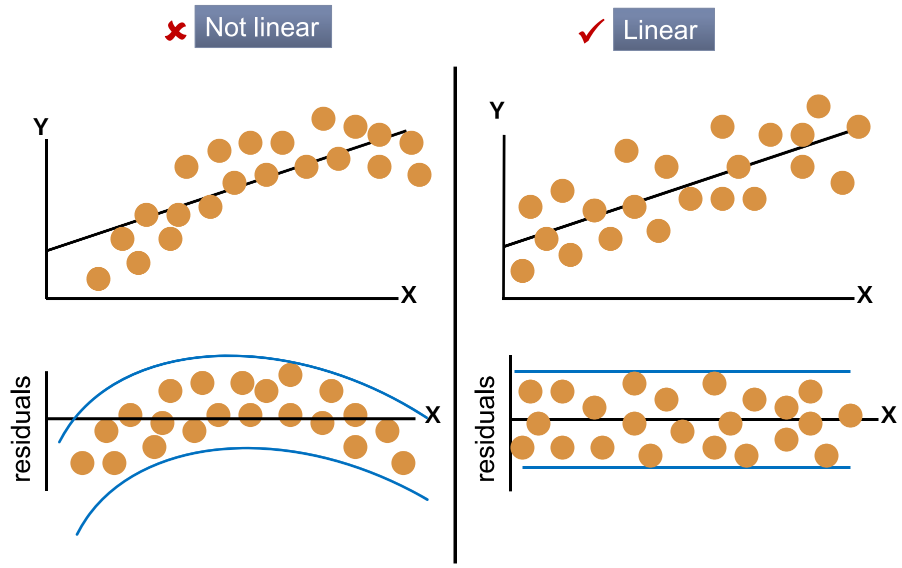
Residual analysis

- The residual for observation i, e_i, is the difference between the observed and predicted values.

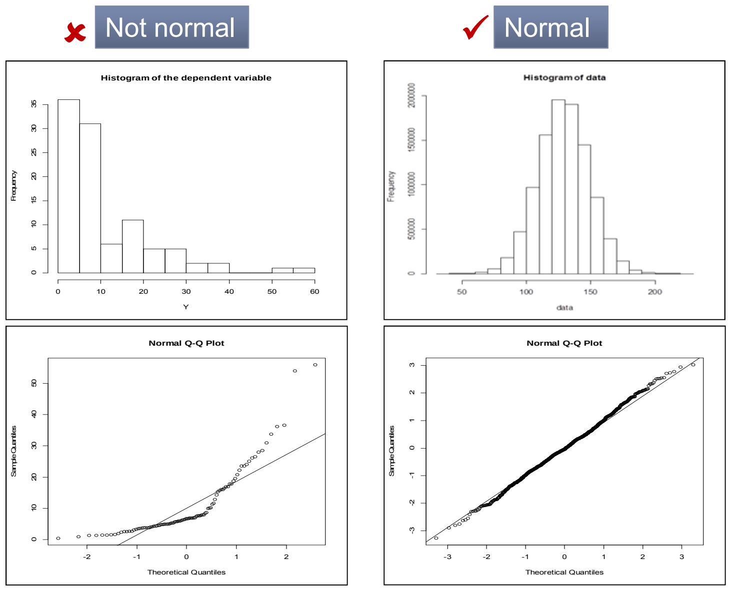
$$e_i = Y_i - \hat{Y}_i$$

- Check the assumptions of regression by examining the residuals.
  - Linearity assumption
  - Independence assumption
  - Normal distribution assumption
  - Constant variance for all levels of X (homoscedasticity).
- Graphical analysis of residuals
  - Plot the residuals against X.

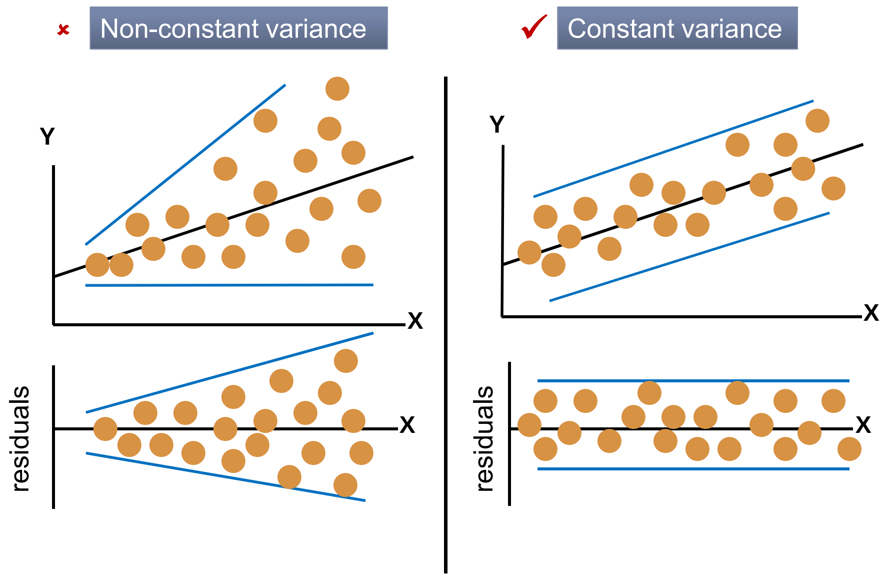
Residual analysis for linearity



Residual analysis for normality



Residual analysis for equal variance



SPSS example: The data

- Suppose we collect data on two variables:
  - Waist to hip ratio (X).
  - Cholesterol (Y).
- For each participant we now have a pair of observations (X_i, Y_i).

ID	Cholesterol (Y)	Ratio (X)
1	203	3.60
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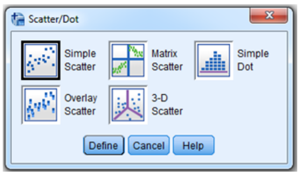
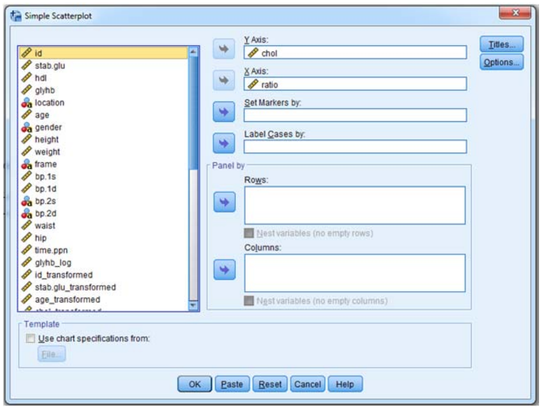
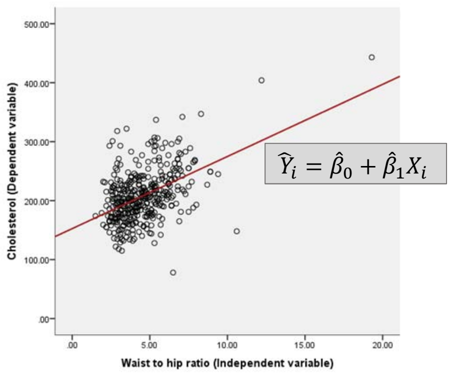
Steps in SPSS: Scatter plot and prediction line

Graphs → Legacy dialogs → Scatter/Dot





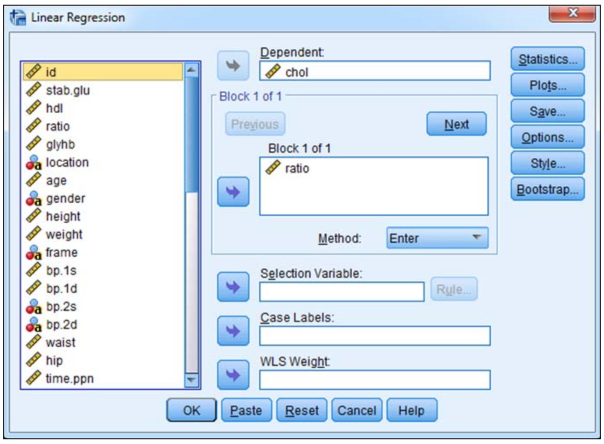
SPSS output: Scatter plot and prediction line



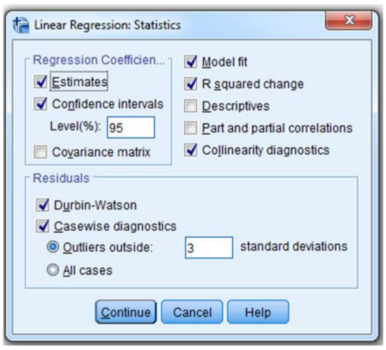
Predicted cholesterol $= 0.970 + 0.387 \times \mathit{Waist}$

Steps in SPSS: Simple linear regression model

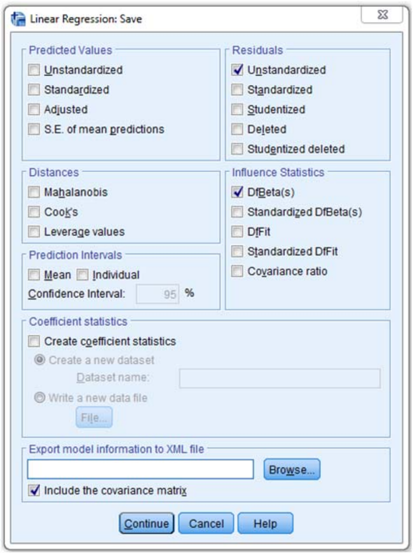
Analyze > Regression > Linear



Steps in SPSS: Statistics tab



Steps in SPSS: Save tab



SPSS output: Regression equation

Coefficients[a]

Model		Unstandardized Coefficients B	Unstandardized Coefficients Std. Error	Standardized Coefficients Beta	t	Sig.	95.0% Confidence Interval for B Lower Bound	95.0% Confidence Interval for B Upper Bound
1	(Constant)	171.233	14.858		11.524	.000	142.022	200.443
	waist	.970	.387	.124	2.503	.013	.208	1.731

a. Dependent Variable: chol

$$\text{Predicted cholesterol} = 171.233 + 0.970 \times Waist$$

SPSS output: R^2 values

Model Summary[b]

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate
1	.124[a]	.015	.013	44.18890

a. Predictors: (Constant), waist

b. Dependent Variable: chol

1.5% of the variance of cholesterol could be explained by waist to hip ratio.

SPSS output: Inferences about the slope

Coefficients[a]

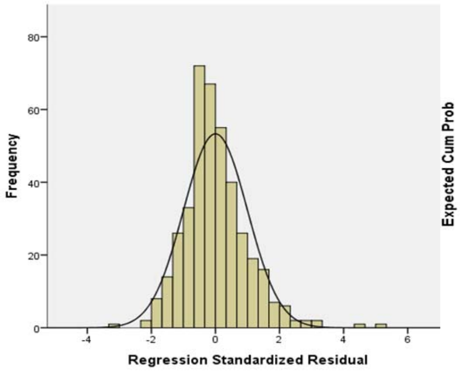
Model		Unstandardized Coefficients B	Unstandardized Coefficients Std. Error	Standardized Coefficients Beta	t	Sig.	95.0% Confidence Interval for B Lower Bound	95.0% Confidence Interval for B Upper Bound
1	(Constant)	171.233	14.858		11.524	.000	142.022	200.443
	waist	.970	.387	.124	2.503	.013	.208	1.731

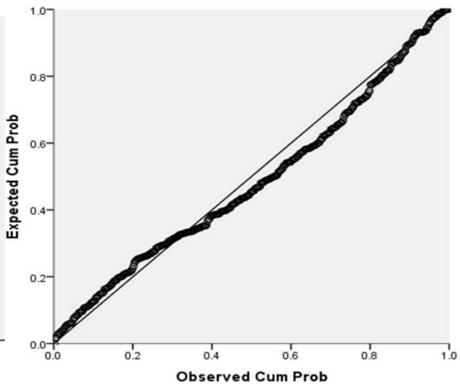
a. Dependent Variable: chol

$$H_0\colon \beta_1 = 0 \qquad H_A\colon \beta_1 \neq 0$$

Waist to hip ratio is a significant predictor of the dependent variable cholesterol, *p*=0.013.

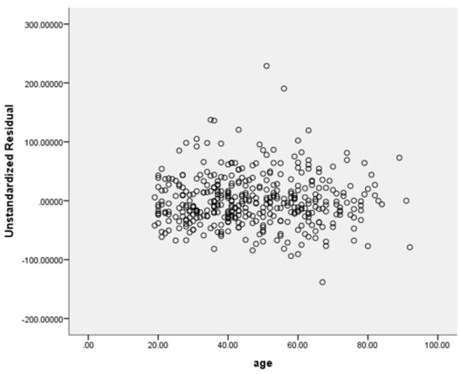
SPSS output: Residual analysis for normality





The residuals are normally distributed.

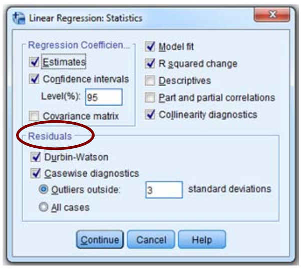
SPSS output: Residual analysis for equal variance



The residuals have constant variance.

SPSS output: Outliers

- If the absolute standardised residual value is larger than 3 then the observation is considered as an outlier.



Casewise Diagnostics[a]

Case Number	Std. Residual	chol	Predicted Value	Residual
4	-3.935	78.00	232.0443	-154.04427
47	-3.428	148.00	282.1938	-134.19385
134	3.320	318.00	188.0105	129.98951
378	3.025	337.00	218.5895	118.41050
381	3.235	322.00	195.3495	126.65055

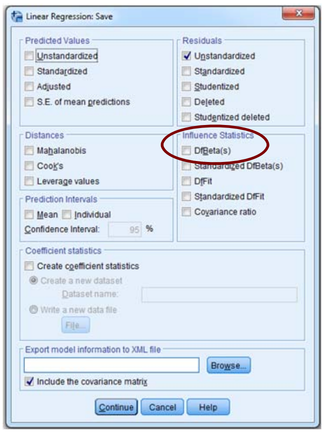
a. Dependent Variable: chol

SPSS output: Influential cases

Analyze → Regression → Linear → Save tab

- ➢ $DFBETAS$: It is the difference between the estimated regression coefficient $\hat{\beta}_k$ based on all n cases and the regression coefficient obtained when the i^{th} case is omitted.

- ➢ If the absolute value of the $DFBETAS$ value exceeds $\frac{2}{\sqrt{n}}$ the value can be viewed as an influential value.



Introduction: Multiple linear regression analysis

- In the preceding chapter, we saw how simple linear regression can be used to explore the nature of the relationship between two variables.
- If knowing the value of a single explanatory variable improves our ability to predict the response, we might expect that additional explanatory variables could be used to our advantage.
- To investigate the more complicated relationship among a number of different variables, we use a natural extension of simple linear regression analysis known as multiple linear regression analysis.

Introduction: Multiple linear regression analysis

- Multiple linear regression equation:

$$\hat{Y} = \hat{B}_0 + \hat{B}_1 X_1 + \hat{B}_2 X_2 + \cdots + \hat{B}_p X_p$$

- The regression coefficients are still estimated by using the method of least squares.
- The independent variables can be continuous or categorical variables.
- In the case of categorical variables we need to use dummy variables.

Adjusted R-squared

- The inclusion of an additional variable in a model can never cause R^2 to decrease.
- To get around this problem, we can use a second measure, called the $adjusted\ R^2$, that compensates for the added complexity of a model.
- The $adjusted\ R^2$ increases when the inclusion of a variable improves our ability to predict the response and decreases when it does not.

Model Summary[b]

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Change Statistics: R Square Change	F Change	df1	df2	Sig. F Change	Durbin-Watson
1	.506[a]	.256	.251	38.30893	.256	45.624	3	397	.000	1.985

a. Predictors: (Constant), weight, age, ratio

b. Dependent Variable: chol

Standardised coefficients

- Unstandardised coefficients
  - The value of the unstandardised coefficient is dependent on the units of measurement of the variables.
  - It is not possible to compare the relative magnitude of coefficients.
- Standardised coefficients
  - The value of the unstandardised coefficient now does not depend on the units of measurement of the variables.
  - It is now possible to compare the relative magnitude of coefficients.
  - How to standardise:

$$\frac{Y_i - \bar{Y}}{sd(Y_i)} \qquad \frac{X_i - \bar{X}}{sd(X_i)}$$

Coefficients[a]

Model		Unstandardized Coefficients: B	Std. Error	Standardized Coefficients: Beta	t	Sig.	95.0% Confidence Interval for B: Lower Bound	Upper Bound	Collinearity Statistics: Tolerance	VIF
1	(Constant)	143.981	10.679		13.482	.000	122.986	164.976		
	ratio	11.908	1.172	.465	10.157	.000	9.603	14.212	.893	1.120
	age	.448	.119	.165	3.756	.000	.213	.682	.967	1.034
	weight	-.060	.050	-.055	-1.210	.227	-.159	.038	.911	1.098

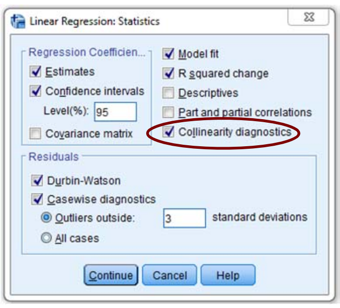
a. Dependent Variable: chol

Multicollinearity

- Multicollinearity exists when there is a strong correlation between two or more predictors (independent variables) in a regression model.
- High levels of collinearity increase the probability that a good predictor of the outcome variable will be found non-significant and rejected from the model (Type II error).
- VIF (variance inflation factor) > 10 indicates a potential problem.
- Tolerance below 0.2 indicates a potential problem.

SPSS output: Multicollinearity

Analyze → Regression → Linear → Statistics



Coefficients[a]

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a. Dependent Variable: chol